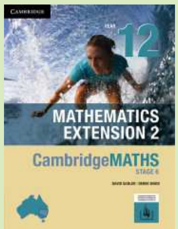
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 2

Topic summary and exercises:

X2 **Further Work with Vectors**

With references to



Name: ....

Initial version by I. Ham, April 2020, with additional contributions from M. Ho. Last updated May 20, 2024.
Various corrections by students & members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under ⓒⓒ CC BY 2.0.

Symbols used

- (!) Beware! Heed warning.
- (A) Mathematics Advanced content.
- (X1) Mathematics Extension 1 content.
- (L) Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argumen

Syllabus subtopics

MEX-V1 Further Work with Vectors

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Extension 2* (Sadler & Ward, 2019) or *Mathematics for Australia 12 Specialist Mathematics* (Haese, Haese, & Humphries, 2017) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Vectors in Three Dimensions

Learning Goal(s)

Knowledge
What are vectors in three dimensions

Skills
Algebraic operations with three-dimensional vectors

Understanding
Able to interpret operations involving three-dimensional vectors geometrically

By the end of this section am I able to:

28.1 Understand and use a variety of notations and representations for vectors in three dimensions

28.2 Perform addition and subtraction of three-dimensional vectors and multiplication of three dimensional vectors by a scalar algebraically and geometrically, and interpret these operations in geometric terms

28.3 Define, calculate and use the magnitude of a vector in three dimensions

28.6 Use Cartesian coordinates in two and three-dimensional space

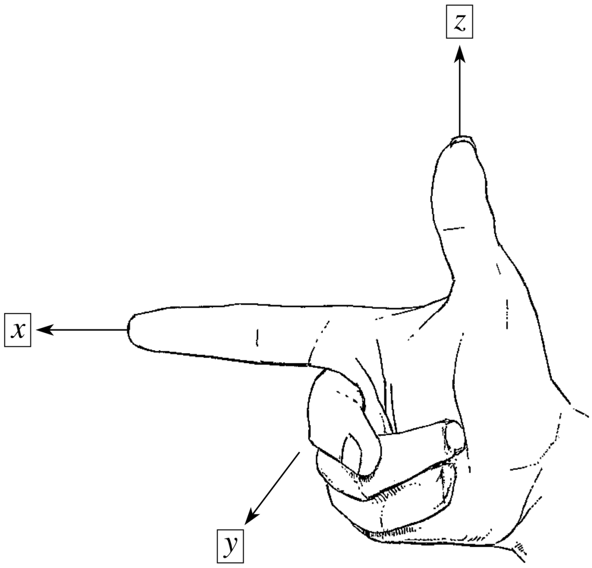
1.1 Cartesian Coordinates in Three Dimensions

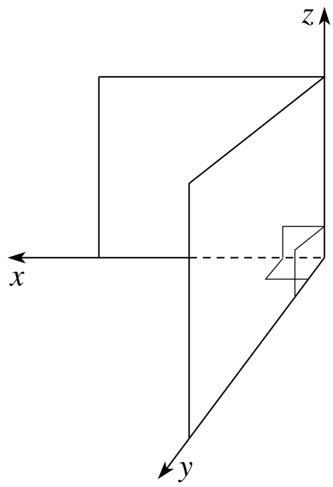
Fill in the spaces

- **The third axis**, called the ...., is required for drawing graphs in three dimensions.
- The .... is .... to both the x axis and y axis.
- The three axes in three dimensions are .... axes.

Important note

The **direction** of the ... . needs to be determined correctly!





Definition 1

Cartesian coordinates in two dimensions

- The two axes divide the plane into ....

Cartesian coordinates in three dimensions

- The xy-plane is the plane containing .... and ....
- The three planes, xy-plane, xz-plane and yz-plane, divide the 3D space into ....

Important note

- Later at university, 2D space will often be denoted $\mathbb{R}^2$ (pronounced "R two"), 3D space will be denoted $\mathbb{R}^3$ (pronounced "R three").
- Some references to $\mathbb{R}^2$ and $\mathbb{R}^3$ will be used throughout this summary.
- Scalars continue to be part of real numbers, e.g. $\lambda \in \mathbb{R}$. Vectors belonging to a particular space will be denoted

$$\underset{\sim}{u} \in \mathbb{R}^2 \quad \text{or} \quad \underset{\sim}{v} \in \mathbb{R}^3$$

as appropriate.

1.2 Vector algebra in $\mathbb{R}^3$

1.2.1 Algebraic representation and operations with vectors

Definition 2

Basis vectors in 3D the .... $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are aligned with the x, y and z axis respectively.

$$\text{i.e. } \underset{\sim}{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underset{\sim}{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \underset{\sim}{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Definition 3

A **vector in three dimensions** can be expressed in a variety of forms:

- ....

$$\overrightarrow{OA} \quad \text{where} \quad A(a, b, c)$$

- .... form:

$$\overrightarrow{OA} = \underset{\sim}{r} = a\,\underset{\sim}{i} + b\,\underset{\sim}{j} + c\,\underset{\sim}{k}$$

- .... notation:

$$\overrightarrow{OA} = \underset{\sim}{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

	2D	3D
	$\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$	
Equality	$\underset{\sim}{u} = \underset{\sim}{v} \Leftrightarrow \begin{cases} u_1 = v_1 \\ u_2 = v_2 \end{cases}$	$\underset{\sim}{u} = \underset{\sim}{v} \Leftrightarrow \begin{cases} u_1 = v_1 \\ u_2 = v_2 \\ u_3 = v_3 \end{cases}$
Zero vector	$\underset{\sim}{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	
Negative	$-\underset{\sim}{u} = \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix}$	$-\underset{\sim}{u} = \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix}$
Vector addition	$\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$	
Scalar multiplication	$k\underset{\sim}{u} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$	

Example 1

Find $2\underset{\sim}{u} - \underset{\sim}{v}$ when $\underset{\sim}{u} = \underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$.

1.2.2 (R) Properties of vectors in space

Important note

The following properties are true for **vectors in 2D**!

Laws/Results

Suppose that $\lambda, \mu \in \mathbb{R}$ and $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ are vectors in 3D space.

- $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$
- $\underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w}) = (\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w}$
- $\lambda(\mu\underset{\sim}{v}) = (\lambda\mu)\underset{\sim}{v}$
- $(\lambda + \mu)\underset{\sim}{v} =$ ....
- $\lambda(\underset{\sim}{u} + \underset{\sim}{v}) =$ ....
- $\left|\lambda\underset{\sim}{u}\right| =$ ....

Steps

The above properties can be easily proven with simple algebra. For example, prove

$$\lambda\left(\underset{\sim}{u} + \underset{\sim}{v}\right) = \lambda\underset{\sim}{u} + \lambda\underset{\sim}{v}$$

for $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$:

1. Let $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\underset{\sim}{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, and λ be a scalar.

2. Fully expand out into column vector notation:

$\lambda\left(\underset{\sim}{u} + \underset{\sim}{v}\right) =$ ....

$=$ .... $=$ ....

$=$ .... $=$ ....

$=$ ....

1.2.3 Ⓡ Parallel vectors

Definition 4

Suppose that $\underset{\sim}{u}$ and $\underset{\sim}{v} \in \mathbb{R}^3$.

- Two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are **parallel** if $\underset{\sim}{v} = \lambda \underset{\sim}{u}$ for some .... - .... number λ.
- They have the same direction if $\lambda > 0$.
- They are **anti-parallel** if $\lambda < 0$.

1.2.4 Magnitude of a vector and unit vector

Example 2

(a) Draw the rectangular block with faces parallel to the three coordinate planes such that $(0, 0, 0)$ and $(3, -2, 4)$ are opposite vertices.

(b) Hence find the magnitude of the vector $\underset{\sim}{u} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$

(c) Find the unit vector having the same direction as $\underset{\sim}{u}$.

(d) Find the magnitude of $\underset{\sim}{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the unit vector having the same direction as $\underset{\sim}{u}$.

(e) Find the distance between $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$.

Definition 5

The magnitude or length of the vector $\underset{\sim}{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is

$$|\underset{\sim}{u}| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Definition 6

If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ are two points in 3D space, then the **magnitude** of $\overrightarrow{AB}$ is

$$\left|\overrightarrow{AB}\right| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which is the **distance** of the points A and B.

Example 3

(Haese et al., 2017, Ex 5D) Find the shortest distance from $Q(3, 1, -2)$ to:

(a) the y-axis. (b) the origin. (c) the y-z plane.

Example 4

[Ex 5A] (Sadler & Ward, 2019) Show that the points $P(1, 0, 0), Q(-3, -1, 1)$ and $R(-2, 3, 4)$ lie on a sphere with centre $C(-1, 1, 2)$.

Example 5

Find a vector $\underset{\sim}{b}$ of length 7 units in the opposite direction to $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Example 6

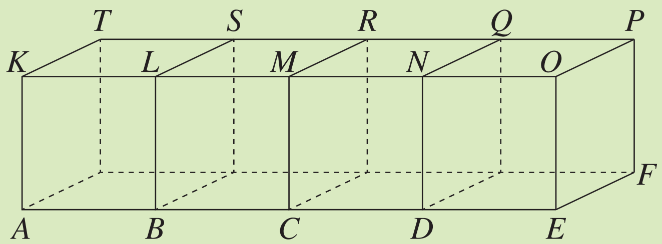
Prove that $A(-1, 2, 3), B(4, 0, -1)$ and $C(14, -4, -9)$ are collinear.

Example 7

Calculate the angle that $\underset{\sim}{a} = \overrightarrow{OA}$ makes with the x-axis, where $A = (2, 3, 4)$. Give your answer correct to the nearest degree.

Example 8

[2021 Ext 2 HSC Q1] For cubes are placed in a line as shown on the diagram.



Which of the following vectors is equal to $\overrightarrow{AB} + \overrightarrow{CQ}$?

(A) $\overrightarrow{AQ}$ (B) $\overrightarrow{CP}$ (C) $\overrightarrow{PB}$ (D) $\overrightarrow{RA}$

Further exercises

Ex 5A (Sadler & Ward, 2019)

- Q1-13

Ex 5B

- Q1-17

Ex 5A-5E (Haese et al., 2017)

- All questions

Section 2

The Dot Product

Learning Goal(s)

Knowledge
What is dot product of three-dimensional vectors

Skills
How to calculate dot product of three-dimensional vectors

Understanding
Able to interpret dot product of three-dimensional vectors algebraically and geometrically

By the end of this section am I able to:
28.4 Define and use the scalar (dot) product of two vectors in three dimensions

2.1 Algebraic representation

Definition 7

Suppose that $\underset{\sim}{u} = \overrightarrow{OU} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\underset{\sim}{v} = \overrightarrow{OV} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Then,

....

2.2 Geometric representation

Definition 8

The angle θ between two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ can be found using

$$\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}|\,|\underset{\sim}{v}|}$$

Steps

Proof

1. Draw situation representing the angle θ between two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$, and vector $\underset{\sim}{v} - \underset{\sim}{u}$.

2. Let $\left|\underset{\sim}{u}\right| = a$, $\left|\underset{\sim}{v}\right| = b$ and $\left|\underset{\sim}{v} - \underset{\sim}{u}\right| = c$. Find a, b and c.

3. Apply the cosine rule to the triangle:

$$\therefore \left|\underset{\sim}{u}\right| \left|\underset{\sim}{v}\right| \cos\theta = \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

4. Consequently,

$$\cos\theta = \ \ldots\ldots\ldots\ldots\ldots$$

2.3 Properties

Laws/Results

- $\underset{\sim}{u} \cdot \underset{\sim}{u} = \left|\underset{\sim}{u}\right|^2$
- $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$
- $\underset{\sim}{u} \cdot (\lambda \underset{\sim}{v}) = \lambda(\underset{\sim}{u} \cdot \underset{\sim}{v})$
- $\underset{\sim}{u} \cdot (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{w}$
- If both $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are non-zero vectors, then

  (i) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0 \ \Leftrightarrow\ \underset{\sim}{u}$ and $\underset{\sim}{v}$ are ……………………………………

  (ii) $\left|\underset{\sim}{u} \cdot \underset{\sim}{v}\right| = \left|\underset{\sim}{u}\right| \left|\underset{\sim}{v}\right| \ \Leftrightarrow\ \underset{\sim}{u}$ and $\underset{\sim}{v}$ are …………………… .

Example 9

Find any values of λ for which $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, where

$$\underset{\sim}{a} = \begin{pmatrix} \lambda \\ 1 \\ 2 \end{pmatrix} \text{ and } \underset{\sim}{b} = \begin{pmatrix} \lambda - 1 \\ 2 \\ -4 \end{pmatrix}$$

Example 10

[Ex 5C] (Sadler & Ward, 2019) The points A, B and C have position vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively relative to the origin O.

If $\overrightarrow{AB} \perp \overrightarrow{OC}$ and $\overrightarrow{BC} \perp \overrightarrow{OA}$, prove that $\overrightarrow{AC} \perp \overrightarrow{OB}$.

Example 11

[Ex 5C] (Sadler & Ward, 2019) Two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are such that $\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$ and $|\underset{\sim}{b}| = |\underset{\sim}{a}|\sqrt{2}$. Show that $2\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$.

2.4 Applications

Example 12

Find the angle at the origin subtended by AB for the points $A = (1, 1, 2)$ and $B = (-2, 3, -1)$. Round the answer to the nearest degree.

2.4.1 Ⓡ Projections in 3D

Important note

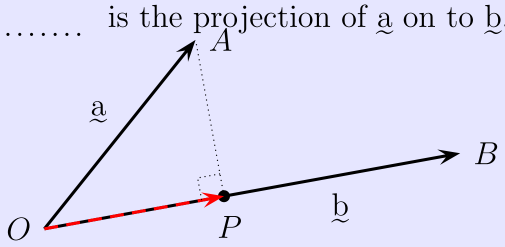
The formula is the same as for **two dimensions**!

Definition 9

Projections in 3D: Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ in three dimensions. The *projection* of a vector $\underset{\sim}{a}$ on another vector $\underset{\sim}{b}$ is a vector parallel to $\underset{\sim}{b}$.

$$\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\,\underset{\sim}{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In the diagram given, ... is the projection of $\underset{\sim}{a}$ on to $\underset{\sim}{b}$.



Assumption: $\underset{\sim}{b}$ is a ....

Steps

Proof

1. $\overrightarrow{OP} = \lambda\,\underset{\sim}{b}$, where λ is a non-zero scalar.

2. Use dot product on $\overrightarrow{OB}$ and $\overrightarrow{PA}$

$$\therefore \text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\,\underset{\sim}{b}$$

Example 13

Find the projection of $\overrightarrow{OA}$ on to $\overrightarrow{OB}$ for $A = (4, 2, -3)$ and $B = (-1, 1, 1)$.

Example 14

Find the perpendicular distance from $P = (2, 1, 0)$ to the line through $A = (-1, 0, 2)$ and $B(1, 1, 3)$.

Further exercises

Ex 5C (Sadler & Ward, 2019)

- Q1-19

Ex 5F (Haese et al., 2017)

- Q1-8

Ex 5G.1, 5G.2

- All questions

Section 3

Ⓡ Vector Proofs in Geometry

Learning Goal(s)

Knowledge
How to prove geometrical results

Skills
Construct proofs logically and coherently

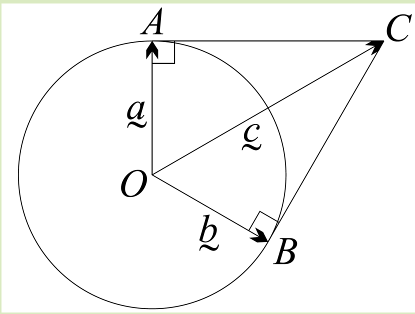
Understanding
How proofs work with three-dimensional vectors

By the end of this section am I able to:

28.5 Prove geometric results in the plane and construct proofs in three dimensions

Example 15

(Sadler & Ward, 2019) Point C is outside a circle with centre O. The points of contact of the two tangents from C to the circle are A and B. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. Prove the following.

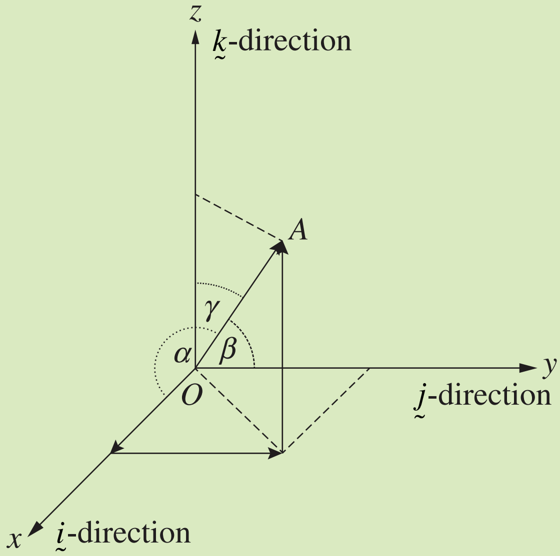


(a) Tangents CA and CB subtend equal angles at the centre O.

(b) $CA = CB$

Example 16

[2020 Ext 2 HSC Sample Q15] (4 marks) The point A has (non-zero) position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and the vector $\overrightarrow{OA}$ makes angles α, β, γ with the x, y and z axes respectively.



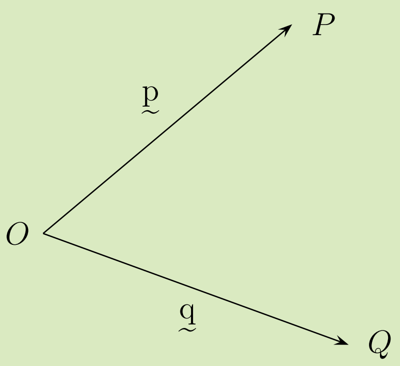
By taking a dot product with the three unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, prove that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Example 17

[2020 NBHS Mathematics Ext 2 Assessment Task 3 Q3]

(a) In the following diagram, let $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$.

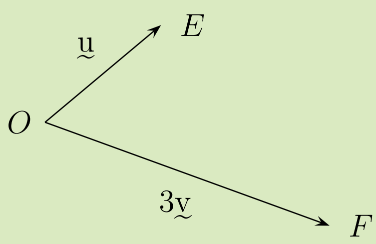


i. Reproduce the diagram on to your writing booklet. In addition, draw the vector **1**

$$\overrightarrow{OR} = \underset{\sim}{p} + \underset{\sim}{q}$$

ii. State the conditions on $\left|\underset{\sim}{p}\right|$ and $\left|\underset{\sim}{q}\right|$ such that $\overrightarrow{OR}$ bisects $\angle POQ$, giving brief reason(s). **2**

(b) The following figure shows $\overrightarrow{OE} = \underset{\sim}{u} = \begin{pmatrix} 2 \\ 5 \\ -7 \end{pmatrix}$ and $\overrightarrow{OF} = 3\underset{\sim}{v} = \begin{pmatrix} 15 \\ 21 \\ 6 \end{pmatrix}$.



i. G is a point such that $\overrightarrow{OG}$ bisects $\angle EOF$ and $\overrightarrow{EG} = \lambda\underset{\sim}{v}$, where $\lambda \in \mathbb{R}$. Find the vector $\overrightarrow{OG}$ in column vector notation. **2**

ii. State the name of the shape formed by $OEGF$, and a brief reason for why $OEGF$ forms this shape. **2**

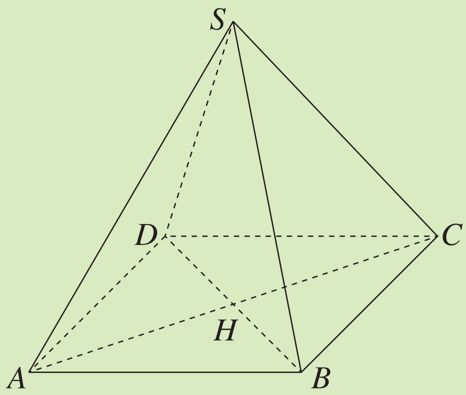
iii. Show that the area A of, $OEGF$ is **3**

$$A = 2\sqrt{5123}$$

Hint: Consider $\text{proj}_{\underset{\sim}{v}}\, \underset{\sim}{u}$.

Example 18

[2021 Ext 2 HSC Q12] The diagram shows the pyramid $ABCDS$ where $ABCD$ is a square. The diagonals of the square bisect each other at H.



i. Show that $\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC} + \overrightarrow{HD} = \underset{\sim}{0}$. **1**

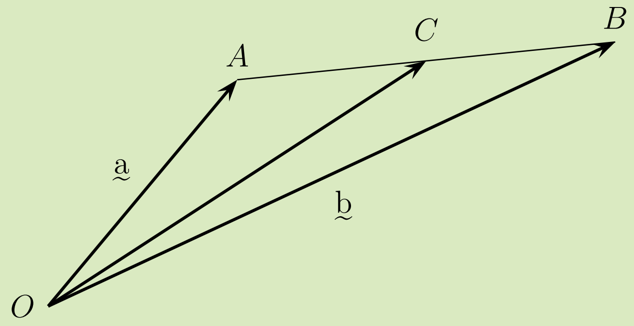
Let G be the point such that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} + \overrightarrow{GS} = \underset{\sim}{0}$.

ii. Using (i), or otherwise, show that $4\overrightarrow{GH} + \overrightarrow{GS} = \underset{\sim}{0}$. **2**

iii. Find the value of λ such that $\overrightarrow{HG} = \lambda\overrightarrow{HS}$. **1**

Example 19

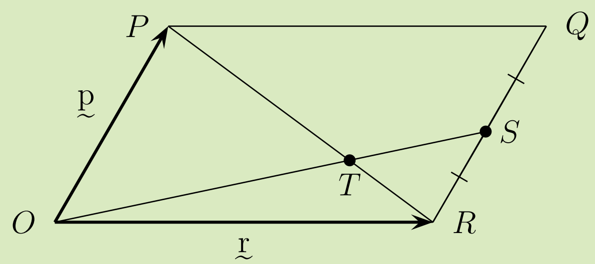
[2020 Ext 2 HSC Q15] The point C divides the interval AB so that $\dfrac{CB}{AC} = \dfrac{m}{n}$. The position vectors of A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, as shown in the diagram.



i. Show that $\overrightarrow{AC} = \dfrac{n}{m+n}(\underset{\sim}{b} - \underset{\sim}{a})$. **2**

ii. Prove that $\overrightarrow{OC} = \dfrac{m}{m+n}\underset{\sim}{a} + \dfrac{n}{m+n}\underset{\sim}{b}$. **1**

Let $OPQR$ be a parallelogram with $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OR} = \underset{\sim}{r}$. The point S is the midpoint of QR and T is the intersection of PR and OS, as shown in the diagram.



iii. Show that $\overrightarrow{OT} = \dfrac{2}{3}\underset{\sim}{r} + \dfrac{1}{3}\underset{\sim}{p}$. **3**

iv. Using parts (ii) and (iii), or otherwise, prove that T is the point that divides the interval PR in the ration 2:1. **1**

Further exercises

Ex 5E (Sadler & Ward, 2019)

- Q1-10

Ex 5H (Haese et al., 2017)

- Q1-4

Section 4

The Vector Equation of a Line

Learning Goal(s)

Knowledge
What is vector equation

Skills
Find vector equation and determine when two lines are parallel, perpendicular or skewed

Understanding
The use of vector equation

By the end of this section am I able to:

28.9 Understand and use the vector equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$ of a straight line through points A and B where R is a point on AB, $\underset{\sim}{a} = \overrightarrow{OA}$, $\underset{\sim}{b} = \overrightarrow{OB}$, λ is a parameter and $\underset{\sim}{r} = \overrightarrow{OR}$.

28.10 Make connections in two dimensions between the equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$ and $y = mx + c$.

28.11 Determine a vector equation of a straight line or straight-line segment, given the position of two points or equivalent information, in two and three dimensions

28.12 Determine when two lines in vector form are parallel

28.13 Determine when intersecting lines are perpendicular in a plane or three dimensions

28.14 Determine when a given point lies on a given line in vector form

4.1 Introduction

Fill in the spaces

- **1-Dimension**: $x = 0$ is a .... on a real number line.
- **2-Dimension**: $x = 0$ is a .... on a 2D x-y plane.
- **3-Dimension**: $x = 0$ is a .... in a 3D space with x, y, z axes as coordinate axes.

Important note

In three dimensions,

- A linear equation with x, y, z with non-zero coefficients represents a ....
- Cartesian form of a line is essentially a system of two linear equations (Two .... planes always form a ....).

Fill in the spaces

- In both 2-dimensional and 3-dimensional geometry, the **equation of a line** can be determined using its .... and any .... on the line.
- On a **2D x-y coordinate plane**, there is only .... line through a fixed point with a certain gradient.
- In **3D space**, is there also only one line through a fixed point with a certain gradient?
  E.g. Is there only one line through the origin with gradient 1? ....

Important note

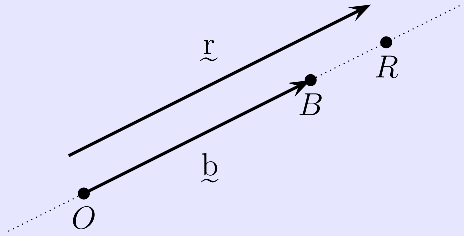
A line in **3D** can be specified by a .... on the line and a vector .... to it.

4.2 Lines in 2 Dimensions

4.2.1 Lines through the origin

Definition 10

Let O be the origin and let B be another point with position vector $\underset{\sim}{b}$. Let R be a variable point in OB with position vector $\underset{\sim}{r}$.



Since $OB \parallel OR$, the **equation of the line** through the origin and another point B with position vector $\underset{\sim}{b}$ has vector equation

$$\underset{\sim}{r} = \lambda \underset{\sim}{b}, \quad \text{where } \lambda \in \mathbb{R}$$

Important note

- The position vector of every point in OB is obtained as λ varies.
- λ is a ....

Example 20

(a) Find the vector equation of the line through the origin and the point $B(2, 3)$.

(b) Write the components of the vector equation found in (a).
(i.e. Write the *parametric equations* of the line).

(c) Find the Cartesian equation

Example 21

Determine the Cartesian equation of the line

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mu \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \quad \text{with } -1 \le \mu \le 1$$

Example 22

The position vector of a body at time t is given by

$$\underset{\sim}{r}(t) = (1-t)\,\underset{\sim}{i} + t^2\,\underset{\sim}{j}, \quad t \ge 0$$

(a) Find the Cartesian equation of the path of the body and state the domain

(b) Sketch the path of the body.

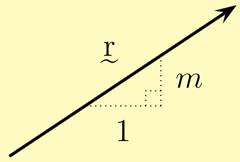
Example 23

Find the vector equations of the x-axis and y-axis.

4.2.2 The direction vector and the gradient

Laws/Results

Consider a vector $\underset{\sim}{r}$ that is parallel to a line with gradient m in 2 dimensions.

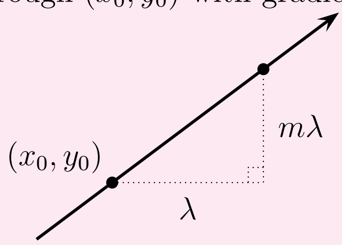


- The vector $\underset{\sim}{r}$ is parallel to ....
- $\begin{pmatrix} 1 \\ m \end{pmatrix}$ is the .... of the line.
- The **vector equation of the line** is ...., where $\lambda \in \mathbb{R}$ is the **parameter**.

4.2.3 Lines through a given point

Steps

Consider a line that goes through (x_0, y_0) with gradient m.



1. The parametric forms of the line are:

2. Equivalently, in **vector form**,

$$\underset{\sim}{r} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix}$$

where $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is the .... of a .... on the line and $\begin{pmatrix} 1 \\ m \end{pmatrix}$ is the ....

Definition 11

In general, the **vector equation** of the line through **a point with position vector** $\underset{\sim}{a}$ with **direction** $\underset{\sim}{b}$ is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

where $\lambda \in \mathbb{R}$.

Example 24

Let ℓ be the line $y = -3x + 4$ in the Cartesian plane.

(a) Find a vector equation of ℓ.

(b) Find another vector equation representing ℓ.

(c) Represent the line ℓ in parametric form in two different ways.

Example 25

Express the line $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ in Cartesian form.

4.3 Lines in 3 Dimensions

Definition 12

- In **3D**, the **vector equation** of the line through $A(a_1, a_2, a_3)$ with position vector $\underset{\sim}{a}$ and parallel to $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

i.e.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where $\lambda \in \mathbb{R}$.

- Equivalently, its **parametric equations** are

$$\begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \\ z = a_3 + \lambda b_3 \end{cases}$$

Important note

A point with position vector $\underset{\sim}{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ **lies on the line** with vector equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$ if and only if **there exists a real value** λ such that $\underset{\sim}{r}_0 = \underset{\sim}{a}_0 + \lambda \underset{\sim}{b}$.

Example 26

Find the vector equation of the line through A parallel with OB, where $A = (-2, -1, 3)$ and $B = (1, 0, 1)$. Then determine whether or not $C = (0, -1, 4)$ is on this line.

Example 27

Consider the points $A(-1, -2, 3)$ and $B(-2, 1, 0)$.

(a) Evaluate $\overrightarrow{AB}$.

(b) Hence determine the vector equation of AB.

Example 28

Consider the following lines

$$\underset{\sim}{r}_1 = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \underset{\sim}{r}_2 = \begin{pmatrix} -5 \\ 2 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

(a) Find the point where the following lines intersect.

(b) Hence show that the two lines form a right angle at the point of intersection.

Example 29

What is the vector equation of the line perpendicular to $2x - 3y + 4 = 0$ which passes through the point $(-5, 6)$?

Example 30

Let $\underset{\sim}{u} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

(a) Find the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$.

(b) Hence find the shortest distance of the point $(2, -1 - 1)$ from the line

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

where $\lambda \in \mathbb{R}$.

4.4 Skew lines

Fill in the spaces

- In **2D**, two non-parallel lines always ....
- Does this hold true in **3D**? ....

Definition 13

In **3D**, the **non-parallel** lines that **do not intersect** are called .... lines.

Example 31

The lines ℓ_1, ℓ_2 and ℓ_3 are given by

$$\ell_1 : \underset{\sim}{r}_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \qquad \ell_2 : \underset{\sim}{r}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ -6 \end{pmatrix}$$

$$\ell_3 : \underset{\sim}{r}_3 = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} + \nu \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

where $\lambda \in \mathbb{R}$

(a) Prove that ℓ_1 and ℓ_2 are parallel.

(b) Prove that ℓ_2 and ℓ_3 are skew.

Further exercises

Ex 5F (Sadler & Ward, 2019)

- Q1-25

Ex 6B, 6C, 6F, 6G.2 (Haese et al., 2017)

- All questions

Section 5

Vector Equations of Circles and Spheres

Learning Goal(s)

Knowledge
Vector equations of circles and spheres

Skills
Find vector equations of circles and spheres

Understanding
The use of parameters in vector equations of circles and spheres

By the end of this section am I able to:

28.7 Recognise and find the equations of spheres

28.8 Use vector equations of curves in two or three dimensions involving a parameter, and determine a corresponding Cartesian equation in the two-dimensional case, where possible

5.1 Equations of circles in two dimensions

Definition 14

The point V with position vector $\underset{\sim}{\mathrm{v}}$ lies on the circle with radius r and centre the origin if

$$\left|\underset{\sim}{\mathrm{v}}\right| = r$$

Example 32

Determine the point on the circle with centre the origin and radius 2 which is closest to the line $2x + 4y - 15 = 0$, using vectors.

Definition 15

Translate the circle $\left|\underset{\sim}{v}\right| = r$ so that the centre is at C with position vector $\underset{\sim}{c}$. Then,

$$\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r$$

where $\underset{\sim}{v}$ is the position vector of a variable point on the circle with centre $\underset{\sim}{c}$ and radius r.

Example 33

The line $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ intersects the circle with centre $\underset{\sim}{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and radius 3 at P and Q. The midpoint of chord PQ is M. Find the coordinates of M.

5.2 Equations of spheres in 3 dimensions

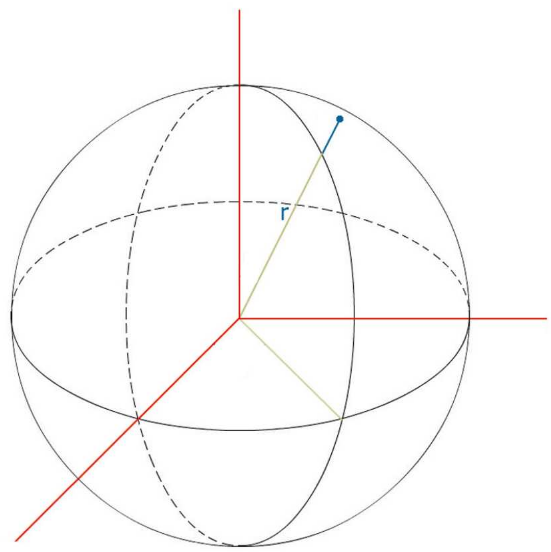
5.2.1 Vector equation of a sphere

Definition 16

- A **sphere** is defined as the set of points in three-dimensional space .... from a .... in space.
- The form of the **vector equation** of a sphere is identical to that of a .... in two dimensions.
- Let $\underset{\sim}{v}$ be the .... of a variable point on the **sphere** with centre $\underset{\sim}{c}$ and radius r. Then,

$$\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r$$

where each vector has .... components.



5.2.2 Cartesian equation of a sphere

Definition 17

- The Cartesian equation of a sphere with centre at the .... and radius r is
$$x^2 + y^2 + z^2 = r^2$$
- The Cartesian equation of a sphere with centre $C(h, k, l)$ and radius r is

....

Steps

1. Let any point on the surface of the sphere with centre at the origin be $V = (x, y, z)$.

2.
$$\because \left|\overrightarrow{OV}\right| = r$$

Example 34

Find the Cartesian equation of the sphere with centre $\underset{\sim}{c} = -\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ which passes through $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$.

Example 35

Find the Cartesian equation of the curve with vector equation

$$\underset{\sim}{v} = (\underset{\sim}{i} + 2\underset{\sim}{j}) + (\cos\theta)\underset{\sim}{i} + (\sin\theta)\underset{\sim}{j}$$

Example 36

Consider the vector equation $\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta$ where

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \underset{\sim}{b} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad \underset{\sim}{c} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

(a) Show that this is a circle by finding its Cartesian equation.

(b) Where on the circle is $\theta = 0$ and in which direction is the circle transversed as θ increases?

Example 37

[**Ex 5G**] (Sadler & Ward, 2019) A line ℓ and a sphere S have equations
$\underset{\sim}{r} = \begin{pmatrix} -3 \\ 16 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -12 \\ 3 \end{pmatrix}$ and $(x-3)^2 + (y+4)^2 + (z+2)^2 = 81$ respectively.

Find the points where ℓ intersects S.

Example 38

[2020 Ext 2 HSC Sample Q15]

i. Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be three 3-dimensional vectors. **1**

Prove that $\underset{\sim}{a} \cdot \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$.

Let $\underset{\sim}{v}$ be the position vector of a point P on a sphere S with centre C and radius r, so that $\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r$, where $\underset{\sim}{c} = \overrightarrow{OC}$. (Do NOT prove this).

ii. The equation of the line ℓ through P in the direction of the vector $\underset{\sim}{m}$ is $\underset{\sim}{w} = \underset{\sim}{v} + \lambda \underset{\sim}{m}$. **2**

Find the values of λ that correspond to the intersection of the line ℓ and the sphere S. Give your answer in terms of $\underset{\sim}{v}$, $\underset{\sim}{c}$ and $\underset{\sim}{m}$.

iii. Deduce that the line ℓ is tangent to the sphere S if and only if **2**

$$\underset{\sim}{m} \cdot \left(\underset{\sim}{v} - \underset{\sim}{c}\right) = 0$$

Interpret this result geometrically.

Example 39

[2021 Ext 2 HSC Q16]

i. The point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O. **2**

Using the position vector of P, $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, and the triangle inequality, or otherwise, show that $|x| + |y| + |z| \geq 1$.

ii. Given the vectors $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that **3**

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$$

iii. As in part (i), the point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O. **2**

Using part (ii), or otherwise, show that $|x| + |y| + |z| \leq \sqrt{3}$.

Further exercises

Ex 5G (Sadler & Ward, 2019)

- Q1-9, 11-16, 18-19

Section 6

Vector Equations of Curves

6.1 Ⓡ Equations of curves in two dimensions

Learning Goal(s)

Knowledge
Projection of three dimensional curves onto a two dimensional plane

Skills
Find Cartesian equation of the projection onto a two dimensional plane

Understanding
Visualise how the projections onto the x-y, y-z and x-z plane determine the shape and direction of the three dimensional curve

By the end of this section am I able to:

28.8 Use vector equations of curves in two or three dimensions involving a parameter, and determine a corresponding Cartesian equation in the two-dimensional case, where possible

Example 40

Sketch the curve with vector equation:

$$\underset{\sim}{r} = \begin{pmatrix} \sin t \\ t \end{pmatrix}$$

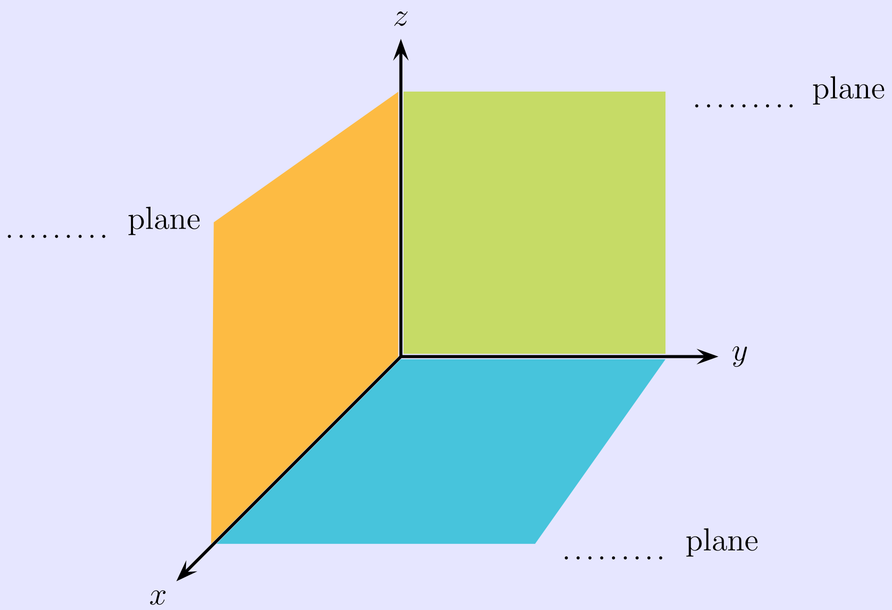
Important note

Hint Find the Cartesian equation with the domain and range for $t \in \mathbb{R}$.

6.2 Projections of three dimensional curves

Definition 18

Curves in three dimensions can be visualised by examining their projection onto the ....



GeoGebra

GeoGebra 3D Calculator

Example 41

Consider the curve with vector equation:

$$\underset{\sim}{r} = \begin{pmatrix} t^2 \\ t^3 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}$$

Sketch the projection of the curve on the:

(a) x-y plane (b) y-z plane (c) x-z plane

Answer: see GeoGebra

Example 42

Consider the curve with vector equation:

$$\underset{\sim}{r} = \begin{pmatrix} t^2 \\ t^4 \\ t^6 \end{pmatrix}, \ t \in \mathbb{R}$$

Sketch the projection of the curve on the:

(a) x-y plane (b) y-z plane (c) x-z plane

Answer: see GeoGebra

Example 43

Consider the two curves with vector equations:

$$\underset{\sim}{r} = \begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix} \text{ and } \underset{\sim}{s} = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}, \; t \in \mathbb{R}$$

For each curve, sketch the projection on the:

(a) x-y plane (b) y-z plane (c) x-z plane

Answer: see GeoGebra: vector $\underset{\sim}{r}$; vector $\underset{\sim}{s}$

Example 44

Consider the curve with vector equation:

$$\underset{\sim}{r} = \begin{pmatrix} t^2 \\ \cos t \\ t \end{pmatrix}, \ t \in \mathbb{R}$$

Sketch the projection of the curve on the:

(a) x-y plane (b) y-z plane (c) x-z plane

Answer: see GeoGebra

Example 45

Consider the curve with vector equation:

$$\underset{\sim}{r} = \begin{pmatrix} t \\ e^t \\ \frac{1}{t} \end{pmatrix}, \ t \in \mathbb{R}$$

Sketch the projection of the curve on the:

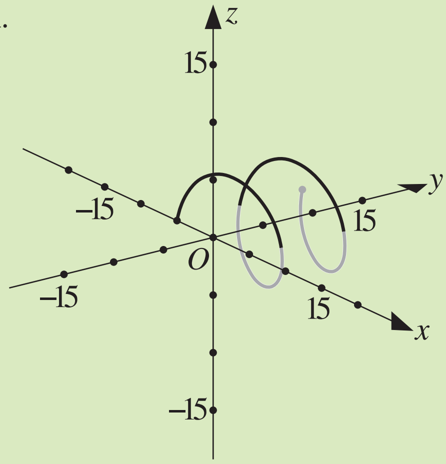
(a) x-y plane (b) y-z plane (c) x-z plane

Answer: see GeoGebra

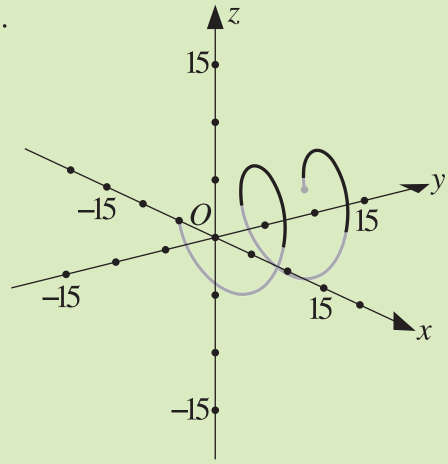
Example 46

[2021 Ext 2 HSC Q7] Which diagram best shows the curve described by the position vector $\underset{\sim}{r}(t) = -5\cos(t)\,\underset{\sim}{i} + 5\sin(t)\,\underset{\sim}{j} + t\,\underset{\sim}{k}$ for $0 \le t \le 4\pi$?

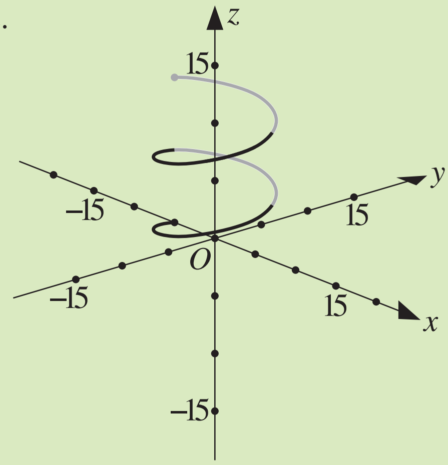
A.



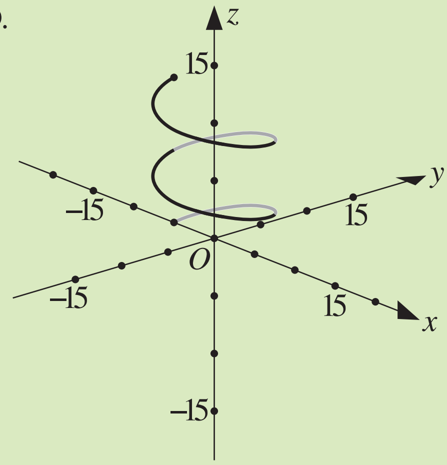
B.



C.



D.



Further exercises

Ex 5G (Sadler & Ward, 2019)

- Q10, 17, 20

Section 7

Past examination questions

- Questions in this section originate from various VCE or WACE papers.
- Questions earmarked (?) indicates that it is uncertain whether a question of this type can appear in the new 2019-2020 syllabuses. It is uncertain due to one, or both of the following:
  - Level of difficulty - does it get this difficult?
  - Reach into other parts of the syllabuses - does it go this far outside of the scope?
- Two additional terms which are not used in the NSW Syllabuses but have equivalents:

Definition 19

Vector resolute is synonymous with the the vector projection.

Definition 20

Scalar projection is the length of the vector projection, with a negative sign if the projection has an opposite direction with respect to $\underset{\sim}{b}$

7.1 2006 VCE Specialist Mathematics

7.1.1 Paper 2 Section 1

16. A unit vector perpendicular to $5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ is

(A) $\frac{1}{4}\left(5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

(B) $2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$

(C) $\frac{1}{29}\left(2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

(D) $\frac{1}{\sqrt{29}}\left(2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}\right)$

(E) $\frac{1}{\sqrt{30}}\left(5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

17. Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$. The angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is

(A) $0°$ (B) $45°$ (C) $30°$ (D) $22.5°$ (E) $90°$

7.2 2007 VCE Specialist Mathematics

7.2.1 Paper 2 Section 1

15. In the cartesian plane, a vector perpendicular to the line $3x + 2y + 1 = 0$ is

(A) $3\underset{\sim}{i} + 2\underset{\sim}{j}$ (C) $2\underset{\sim}{i} - 3\underset{\sim}{j}$ (E) $2\underset{\sim}{i} + 3\underset{\sim}{j}$

(B) $-\frac{1}{2}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j}$ (D) $\frac{1}{2}\underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j}$

17. The angle between the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ is best represented by

(A) $-\frac{4}{9}$ (C) $\pi + \cos^{-1}\left(-\frac{4}{9}\right)$ (E) $\cos^{-1}\left(\pi - \frac{4}{9}\right)$

(B) $-\cos^{-1}\left(\frac{4}{9}\right)$ (D) $\pi - \cos^{-1}\left(\frac{4}{9}\right)$

18. Let $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{v} = a\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$. If the scalar resolute of $\underset{\sim}{v}$ in the direction of $\underset{\sim}{u}$ is 1, then the value of $\underset{\sim}{a}$ is

(A) $-\frac{3}{2}$ (B) $-\frac{2}{3}$ (C) 3 (D) $\frac{2}{3}$ (E) $\frac{3}{2}$

7.3 2008 VCE Specialist Mathematics

7.3.1 Paper 1

Question 8

The coordinates of three points are $A(1, 0, 5)$, $B(-1, 2, 4)$ and $C(3, 5, 2)$.

(a) Express the vector $\overrightarrow{AB}$ in the form $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$. **1**

(b) Find the coordinates of the point D such that $ABCD$ is a parallelogram. **2**

(c) Prove that $ABCD$ is a rectangle. **1**

7.3.2 Paper 2 Section 1

14. If the vectors $\underset{\sim}{a} = m\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m\underset{\sim}{i} + m\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular, then

(A) $m = 0$ (C) $m = -2$ or $m = 6$ (E) $m = -1$ or $m = 1$

(B) $m = -6$ or $m = 2$ (D) $m = -2$ or $m = 0$

7.3.3 Paper 2 Section 2

Question 3

The position vector $\underset{\sim}{r}(t)$ of the front of a toy train at time t seconds on a closed track is given by

$$\underset{\sim}{r}(t) = \sin\left(\frac{t}{3}\right)\underset{\sim}{i} + \frac{1}{2}\sin\left(\frac{2t}{3}\right)\underset{\sim}{j}, \quad t \geq 0$$

where displacement components are measured in metres.

(a) If the front of the train is at the point $P(x, y)$ at time t, show that **1**

$$y^2 = \sin^2\left(\frac{t}{3}\right)\cos^2\left(\frac{t}{3}\right)$$

(b) Hence, find the cartesian equation of the path of the train. **1**

7.4 2009 VCE Specialist Mathematics

7.4.1 Paper 1

Question 3

Resolve the vector $5\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$ into two vector components, one which is parallel to the vector $-2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and one which is perpendicular to it.

7.4.2 Paper 2 Section 1

16. Consider the three vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -3\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = 13\underset{\sim}{i} + 10\underset{\sim}{j} + \underset{\sim}{k}$. It follows that

(A) $\underset{\sim}{c}$ and $\underset{\sim}{b}$ are perpendicular to $\underset{\sim}{a}$

(B) $\underset{\sim}{c}$ is only perpendicular to $\underset{\sim}{b}$

(C) $\underset{\sim}{c}$ is only perpendicular to $\underset{\sim}{a}$

(D) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular to $\underset{\sim}{c}$

(E) $\underset{\sim}{a}$ is only perpendicular to $\underset{\sim}{b}$

7.5 2010 VCE Specialist Mathematics

7.5.1 Paper 1

Question 3

Relative to an origin O, point A has cartesian coordinates $(1, 2, 2)$ and point B has cartesian coordinates $(-1, 3, 4)$.

(a) Find an expression for the vector $\overrightarrow{AB}$ in the form $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$. **1**

(b) Show that the cosine of the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ is $\dfrac{4}{9}$. **1**

(c) Hence, find the exact area of the triangle OAB. **3**

7.5.2 Paper 2 Section 1

15. The scalar resolute of $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ is

(A) $\dfrac{8}{\sqrt{10}}$ (B) $\dfrac{8}{9}(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k})$ (D) $\dfrac{4}{5}(3\underset{\sim}{i} - \underset{\sim}{k})$

(C) 8 (E) $\dfrac{8}{3}$

16. The square of the magnitude of the vector $\underset{\sim}{d} = 5\underset{\sim}{i} - \underset{\sim}{j} + \sqrt{10}\,\underset{\sim}{k}$ is

(A) 6 (B) 34 (C) 36 (D) 51.3 (E) $\sqrt{34}$

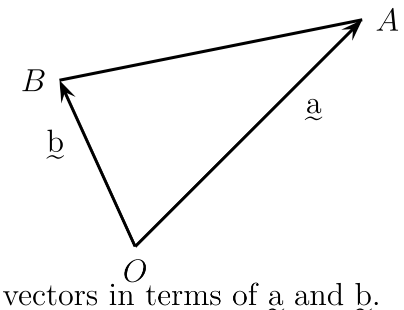
17. The angle between the vectors $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j}$ is exactly

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{3}$ (D) $\dfrac{\pi}{2}$ (E) π

7.5.3 Paper 2 Section 2

Question 1

The diagram below shows a triangle with vertices O, A and B. Let O be the origin, with vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



(a) Find the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

i. $\overrightarrow{MA}$, where M is the midpoint of the line segment OA. **1**

ii. $\overrightarrow{BA}$. **1**

iii. $\overrightarrow{AQ}$, where Q is the midpoint of the line segment AB. **1**

(b) Let N be the midpoint of the line segment OB. Use a vector method to prove that the quadrilateral $MNQA$ is a parallelogram. **3**

Now consider the **particular** triangle OAB with $\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$ and $\overrightarrow{OB} = \alpha\underset{\sim}{i}$ where α, which is greater than zero, is chosen so that the $\triangle OAB$ is isosceles, with $\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA}\right|$.

(c) Show that $\alpha = 4$. **1**

(d) i. Find $\overrightarrow{OQ}$, where Q is the midpoint of the line segment AB. **1**

ii. Use a vector method to show that $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$. **3**

7.6 2011 VCE Specialist Mathematics

7.6.1 Paper 1

Question 9

Consider the three vectors

$$\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} \qquad \underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + m\underset{\sim}{k} \qquad \text{and} \qquad \underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$$

where $m \in \mathbb{R}$.

(a) Find the value(s) of m for which $\left|\underset{\sim}{b}\right| = 2\sqrt{3}$. **2**

(b) Find the value of m such that $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$. **1**

7.6.2 Paper 2 Section 1

12. The angle between the vectors $3\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}$ and $2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, correct to the nearest tenth of a degree, is

(A) $2.0°$ (B) $91.0°$ (C) $112.4°$ (D) $121.3°$ (E) $124.9°$

7.7 2012 VCE Specialist Mathematics

7.7.1 Paper 2 Section 1

15. The vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m^2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ are perpendicular for

(A) $m = -\frac{2}{3}$ and $m = 1$

(B) $m = -\frac{3}{2}$ and $m = 1$

(C) $m = \frac{2}{3}$ and $m = -1$

(D) $m = \frac{3}{2}$ and $m = -1$

(E) $m = 3$ and $m = -1$

7.8 2013 VCE Specialist Mathematics

7.8.1 Paper 1

Question 3

The coordinates of three points are $A(-1, 2, 4), B(1, 0, 5)$ and $C(3, 5, 2)$.

(a) Find $\overrightarrow{AB}$. **1**

(b) The points A, B and C are the vertices of a triangle. Prove that the triangle has a right angle at A. **2**

(c) Find the length of the hypotenuse of the triangle. **1**

7.8.2 Paper 2 Section 1

14. The distance from the origin to the point $P\left(7, -1, 5\sqrt{2}\right)$ is

(A) $7\sqrt{2}$ (B) 10 (C) $6 + 5\sqrt{2}$ (D) 100 (E) $5\sqrt{6}$

15. Let $\underset{\sim}{u} = 4\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = 3\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{w} = -4\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$. Which one of the following statements is **not** true?

(A) $|\underset{\sim}{u}| = |\underset{\sim}{v}|$

(B) $|\underset{\sim}{u}| = |-\underset{\sim}{w}|$

(C) $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly dependent

(D) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

(E) $(\underset{\sim}{u} + \underset{\sim}{w}) \cdot \underset{\sim}{v} = 12$

Note: A set of vectors is said to be *linearly dependent* if at least one of the vectors in the set can be defined as a linear combination of the others, i.e. If $r_1\underset{\sim}{u} + r_2\underset{\sim}{v} + r_3\underset{\sim}{w} = 0$ for some $r_1, r_2, r_3 \in \mathbb{R}$, where at least one of r_1, r_2, r_3 is non-zero.

7.8.3 Paper 2 Section 2

Question 4

Let $\underset{\sim}{a} = -\dfrac{7\sqrt{3}}{3}\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k}$.

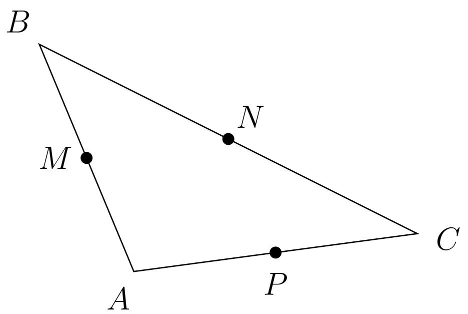
(a) Find a unit vector in the direction of $\underset{\sim}{b}$. **1**

(b) Resolve $\underset{\sim}{a}$ into two vector components, one that is parallel to $\underset{\sim}{b}$ and one that is perpendicular to $\underset{\sim}{b}$. **3**

(c) Find the value of $\underset{\sim}{m}$ such that $\underset{\sim}{c} = m\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ makes an angle of $\dfrac{2\pi}{3}$ with $\underset{\sim}{b}$ and where $\underset{\sim}{c} \neq \underset{\sim}{a}$. **2**

(d) Find the angle, in degrees, that $\underset{\sim}{c}$ makes with $\underset{\sim}{a}$, correct to one decimal place. **2**

(e) For the triangle ABC shown below, the midpoints of the sides are the points M, N and P. Let $\overrightarrow{AC} = \underset{\sim}{u}$ and $\overrightarrow{CB} = \underset{\sim}{v}$.



i. Express $\overrightarrow{AN}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **1**

ii. Express $\overrightarrow{CM}$ and $\overrightarrow{BP}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **2**

iii. Hence, simplify the expression $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP}$. **1**

7.9 2014 VCE Specialist Mathematics

7.9.1 Paper 1

Question 1

Consider the vector $\sqrt{3}\,\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\,\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the x, y and z axes respectively.

(a) Find the unit vector in the direction of $\underset{\sim}{a}$. **1**

(b) Find the acute angle that $\underset{\sim}{a}$ makes with the positive direction of the x-axis. **2**

(c) The vector $\underset{\sim}{b} = 2\sqrt{3}\,\underset{\sim}{i} + m\,\underset{\sim}{j} - 5\,\underset{\sim}{k}$. Given that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, find the value of $\underset{\sim}{m}$. **2**

7.9.2 Paper 2 Section 1

15. If θ is the angle between $\underset{\sim}{a} = \sqrt{3}\,\underset{\sim}{i} + 4\,\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\,\underset{\sim}{j} + \sqrt{3}\,\underset{\sim}{k}$, then $\cos 2\theta$ is **1**

(A) $-\frac{4}{5}$ (B) $\frac{7}{25}$ (C) $-\frac{7}{25}$ (D) $\frac{14}{25}$ (E) $-\frac{24}{25}$

16. Two vectors are given by $\underset{\sim}{a} = 4\,\underset{\sim}{i} + m\,\underset{\sim}{j} - 3\,\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\,\underset{\sim}{i} + n\,\underset{\sim}{j} - \underset{\sim}{k}$, where $m, n \in \mathbb{R}^+$. If $\left|\underset{\sim}{a}\right| = 10$ and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$, then m and n respectively are **1**

(A) $5\sqrt{3},\ \frac{\sqrt{3}}{3}$

(B) $5\sqrt{3},\ \sqrt{3}$

(C) $-5\sqrt{3},\ \sqrt{3}$

(D) $\sqrt{93},\ \frac{5\sqrt{93}}{93}$

(E) $5, 1$

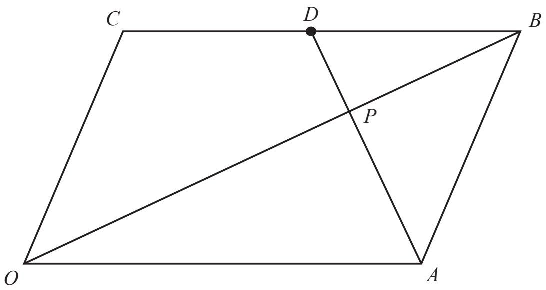
7.9.3 Paper 2 Section 2

Question 3

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$.

(a) Express $\underset{\sim}{a}$ as the **sum** of two vector resolutes, one of which is parallel to $\underset{\sim}{b}$ and the other of which is perpendicular to $\underset{\sim}{b}$. Identify clearly the parallel vector resolute and the perpendicular vector resolute. **5**

(b) $OABC$ is a paralleogram where D is the midpoint of CB. OB and AD intersect at point P. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



i. Given that $\overrightarrow{AP} = \alpha\overrightarrow{AD}$, write an expression for $\overrightarrow{AP}$ in terms of $\alpha, \underset{\sim}{a}$ and $\underset{\sim}{c}$. **2**

ii. Given that $\overrightarrow{OP} = \beta\overrightarrow{OB}$, write another expression for $\overrightarrow{AP}$ in terms of β, $\underset{\sim}{a}$ and $\underset{\sim}{c}$. **1**

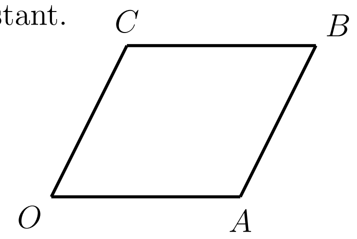
iii. **Hence** deduce the values of α and β. **2**

7.10 2015 VCE Specialist Mathematics

7.10.1 Paper 1

Question 1

Consider the rhombus $OABC$ shown below, where $\overrightarrow{OA} = a\,\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, and a is a positive real constant.



(a) Find a. **1**

(b) Show that the diagonals of the rhombus $OABC$ are perpendicular. **2**

7.10.2 Paper 2 Section 1

17. Points A, B and C have position vectors $\underset{\sim}{a} = 2\underset{\sim}{i}+\underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i}-\underset{\sim}{j}$ and $\underset{\sim}{c} = -3\underset{\sim}{j}+\underset{\sim}{k}$ respectively. The cosine of angle ABC is equal to

(A) $\dfrac{5}{\sqrt{6}\sqrt{10}}$ (B) $\dfrac{7}{\sqrt{6}\sqrt{13}}$ (C) $-\dfrac{1}{\sqrt{6}\sqrt{13}}$ (D) $-\dfrac{7}{\sqrt{21}\sqrt{6}}$ (E) $-\dfrac{2}{\sqrt{6}\sqrt{13}}$

7.10.3 Paper 2 Section 2

Question 4

The position vector $\underset{\sim}{r}(t)$, from origin O, of a model helicopter t seconds after leaving the ground is given by

$$\underset{\sim}{r}(t) = \left(50 + 25\cos\frac{\pi t}{30}\right)\underset{\sim}{i} + \left(50 + 25\sin\frac{\pi t}{30}\right)\underset{\sim}{j} + \frac{2t}{5}\underset{\sim}{k}$$

where $\underset{\sim}{i}$ is a unit vector to the east, $\underset{\sim}{j}$ is a unit vector to the north and $\underset{\sim}{k}$ is a unit vector vertically up. Displacement components are measured in metres.

(a) Find in time, in seconds, required for the helicopter to gain an altitude of 60m. **1**

(b) Find the angle of elevation from O of the helicopter when it is at an altitude of 60m. Give your answer in degrees, correct to the nearest degree. **2**

(c) After how many seconds will the helicopter first be directly above the point of take-off? **1**

(d) (?) Show that the velocity of the helicopter is perpendicular to its acceleration. **3**

(e) (?) Find the speed of the helicopter in ms^{-1}, giving your answer correct to two decimal places. **2**

(f) A treetop has position vector $\underset{\sim}{r} = 60\underset{\sim}{i} + 40\underset{\sim}{j} + 8\underset{\sim}{k}$. Find the distance of the helicopter from the treetop after it has been travelling for 45 seconds. Give your answer in metres, correct to one decimal place. **3**

7.11 2016 VCE Specialist Mathematics

7.11.1 Paper 2 Section 1

11. Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \alpha\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - \underset{\sim}{j} + \alpha^2\underset{\sim}{k}$, where α is a real constant. If the scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\dfrac{74}{\sqrt{273}}$, then α equals **1**

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

12. If $\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -m\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, where m is a real constant, the vector $\underset{\sim}{a} - \underset{\sim}{b}$ will be perpendicular to vector $\underset{\sim}{b}$ where m equals **1**

(A) 0 only (B) 2 only (C) 0 or 2 (D) 4.5 (E) 0 or -2

7.12 2016 WACE Mathematics Specialist

7.12.1 Calculator free

Question 7

Points A and B have respective position vectors $\begin{pmatrix}4\\0\\3\end{pmatrix}$ and $\begin{pmatrix}0\\-2\\5\end{pmatrix}$.

(a) Determine the vector equation for the sphere that has $\overrightarrow{AB}$ as its diameter. **3**

(b) If point O is the origin, consider the plane that contains the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$. **4**

Determine the vector equation for this plane in the form

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = c$$

7.13 2017 VCE Specialist Mathematics

7.13.1 Paper 1

Question 5

Relative to a fixed origin, the points B, C and D are defined respectively by the position vectors $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}, \underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{d} = a\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, where a is a real constant. **4**

Given that the magnitude of $\angle BCD$ is $\frac{\pi}{3}$, find a.

7.14 2018 VCE Specialist Mathematics

7.14.1 Paper 2 Section 1

14. The scalar projection of $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ is **1**

(A) $-\frac{9\sqrt{13}}{13}$

(B) $-\frac{9}{14}(-\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

(C) $-\frac{9\sqrt{14}}{14}$

(D) $-\frac{9}{13}\left(3\underset{\sim}{i} - 2\underset{\sim}{k}\right)$

(E) $-\frac{\sqrt{14}}{2}$

7.15 2019 VCE Specialist Mathematics

7.15.1 Paper 2 Section 1

12. The vector projection of $\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$ in the direction of $m\underset{\sim}{i}+n\underset{\sim}{j}+p\underset{\sim}{k}$ is $2\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k}$, where m, n and p are real constants. The values of m, n and p can be found by solving the equations **1**

(A) $\dfrac{m(m+n-p)}{m^2+n^2+p^2}=2$, $\dfrac{n(m+n-p)}{m^2+n^2+p^2}=-3$ and $\dfrac{p(m+n-p)}{m^2+n^2+p^2}=1$

(B) $\dfrac{m(m+n-p)}{m^2+n^2+p^2}=1$, $\dfrac{n(m+n-p)}{m^2+n^2+p^2}=1$ and $\dfrac{p(m+n-p)}{m^2+n^2+p^2}=-1$

(C) $m+n-p=6$, $m+n-p=-9$ and $m+n-p=-3$

(D) $m+n-p=3m$, $m+n-p=3n$ and $m+n-p=-3p$

(E) $m+n-p=2\sqrt{3}$, $m+n-p=-3\sqrt{3}$ and $m+n-p=\sqrt{3}$

7.15.2 Paper 2 Section 2

Question 4

The base of a pyramid is the parallelogram $ABCD$ with vertices at points $A(2,-1,3)$, $B(4,-2,1)$, $C(a,b,c)$ and $D(4,3,-1)$. The apex (top) of the pyramid is located at $P(4,-4,9)$.

(a) Find the values of a, b and c. **2**

(b) Find the cosine of the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$. **2**

(c) Find the area of the base of the pyramid. **2**

(d) Show that $6\underset{\sim}{i}+2\underset{\sim}{j}+5\underset{\sim}{k}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AD}$, and hence find a unit vector that is perpendicular to the base of the pyramid. **3**

(e) Find the volume of the pyramid. **2**

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \; r \neq 1$$

$$S = \frac{a}{1 - r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$

$A = \dfrac{1}{2}ab\sin C$

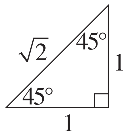
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

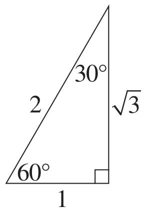
$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$





Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

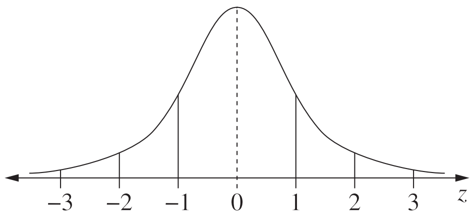
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

References

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